

Bundle gerbes

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Abstract. Just as \mathbf{C}^\times principal bundles provide a geometric realisation of two-dimensional integral cohomology; gerbes or sheaves of groupoids, provide a geometric realisation of three dimensional integral cohomology through their Dixmier-Douady class. I consider an alternative, related, geometric realisation of three dimensional cohomology called a bundle gerbe. Every bundle gerbe gives rise to a gerbe and most of the well-known examples examples of gerbes are bundle gerbes. I discuss the properties of bundle gerbes, in particular bundle gerbe connections and curvature and their associated Dixmier-Douady class.

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1. Introduction

In [1] Brylinski describes Giraud's theory of gerbes. Loosely speaking a gerbe over a manifold M is a sheaf of groupoids over M . Gerbes, via their Dixmier-Douady class, provide a geometric realisation of the elements of $H^3(M, \mathbf{Z})$ analogous to the way that line bundles provide, via their Chern class, a geometric realisation of the elements of $H^2(M, \mathbf{Z})$.

I want to introduce here another sort of object, which is not a sheaf, and which also gives rise to elements of $H^3(M, \mathbf{Z})$. For want of a better name I have called these objects bundle gerbes. A bundle gerbe over M is a pair consisting of a fibration $Y \rightarrow M$ and a principal \mathbf{C}^\times bundle P over the fibre product $Y^{[2]}$. The bundle P is required to have a product, that is a \mathbf{C}^\times bundle morphism which on fibres is of the form:

$$P_{(x,y)} \otimes P_{(y,z)} \rightarrow P_{(x,z)}, \quad (1.1)$$

for any x, y, z in the same fibre of $Y \rightarrow M$. From a bundle gerbe it is possible to construct a presheaf of groupoids and hence a gerbe. However not all gerbes arise in this way.

By considering a connection on P compatible with the product (1.1) it is possible to construct a closed, integral three-form on M analogous to the curvature of a line bundle. This three form is a representative for the image in $H^3(M, \mathbf{R})$ of the Dixmier-Douady class of the corresponding gerbe. Moreover every integral three class is represented by a closed, integral three form arising from a bundle gerbe $P \rightarrow Y^{[2]}$ where $Y \rightarrow M$ is the path fibration. This construction is analogous to the construction of a line bundle with given curvature two-form. It follows from the results in [1] that every isomorphism class of gerbes contains a bundle gerbe. If $Q \rightarrow Y$ is a \mathbf{C}^\times bundle there is a bundle gerbe whose fibre at (x, y) is $\text{Aut}_{\mathbf{C}^\times}(Q_x, Q_y)$. I shall call such a bundle gerbe trivial. The geometric interpretation of the Dixmier-Douady class of a bundle gerbe is that it is the obstruction to the bundle gerbe being trivial.

Many interesting gerbes are bundle gerbes. For instance if $\mathbf{C}^\times \rightarrow \hat{G} \rightarrow G$ is a central extension of groups it is well known that the obstruction to lifting a principal G bundle over M to a principal \hat{G} bundle is a class in $H^3(M, \mathbf{Z})$ and Brylinski discusses the gerbe defined by such a principal bundle whose Dixmier-Douady class is this obstruction. This principal bundle also gives rise to a bundle gerbe in a natural way and this bundle gerbe is trivial precisely when the original bundle lifts to \hat{G} .

Having outlined the virtues of bundle gerbes I should mention two of their deficiencies. The first is that there are many bundle gerbes which are isomorphic as gerbes but not as bundle gerbes. As a consequence the theorem that isomorphism classes of gerbes are in bijective correspondence with $H^3(M, \mathbf{Z})$ does not hold for bundle gerbes. The second is that if a bundle gerbe $P \rightarrow Y^{[2]}$ is non-trivial then the fibres of $Y \rightarrow M$ must be infinite dimensional.

In outline the paper is as follows. Section 2 reviews the properties of \mathbf{C}^\times groupoids and their relationship with line bundles. This is in preparation for Section 3 which introduces bundle gerbes. The motivating example of the lifting of a principal bundle for a central extension is discussed in Section 4 and this leads to the introduction of the Dixmier-Douady class of a bundle gerbe in Section 5. A de Rham representative for this class is provided by the theory of bundle gerbe connections and curvature introduced in Sections 6, 7 and 8. The relationship with gerbes is discussed in Section 9 as in the relationship of the bundle gerbe connection and curvature with Brylinski's connective structure and curving. In Section 10 I show how to construct a bundle gerbe with given Dixmier-Douady class. Section 11 considers the Deligne cohomology class defined by a bundle gerbe with connection and in Section 12 the holonomy of a bundle gerbe connection over a two-sphere is defined. Finally in Section 13 I explain why the fibering $Y \rightarrow M$ has to have infinite dimensional fibres if the bundle gerbe is non-trivial.

I will assume, when talking about gerbes, some familiarity with Brylinski's book [1]. The material on bundle gerbes however is intended to be self-contained.

2. \mathbf{C}^\times groupoids

Denote by \mathbf{C}^\times the group of non-zero complex numbers. If P and Q are principal \mathbf{C}^\times bundles over a manifold Z then it is possible to define a new principal \mathbf{C}^\times bundle $P \otimes Q$ over $Z \times Z = Z^2$. This is called the contracted product [1]. If P and Q are the frame bundles of line bundle L and J respectively then $P \otimes Q$ is the frame bundle of the line bundle $L \otimes J$. To define $P \otimes Q$ take $P \times Q$ which is a principal $\mathbf{C}^\times \times \mathbf{C}^\times$ bundle over Z^2 and quotient by the 'anti-diagonal' copy of \mathbf{C}^\times inside $\mathbf{C}^\times \times \mathbf{C}^\times$, that is, the subgroup of all pairs (z, z^{-1}) . What makes this construction possible, of course, is the fact that \mathbf{C}^\times is abelian. Consider now a manifold X and inside $X^2 \times X^2$ define

$$X^2 \circ X^2 = \{((x, y), (y, z)) \mid x, y, z \in X\}. \quad (2.1)$$

If P is a principal \mathbf{C}^\times bundle over X^2 define $P \circ P$ to be the restriction of $P \otimes P$ to $X^2 \circ X^2$.

Recall [2] that a groupoid is a category with every morphism invertible. Let us consider an equivalent definition that we will see below is easily generalised to define bundle gerbes. Define a \mathbf{C}^\times groupoid to be a principal \mathbf{C}^\times bundle P over X^2 with a product, that is, a \mathbf{C}^\times bundle morphism $P \circ P \rightarrow P$, $(p, q) \mapsto pq$, covering the map $((x, y), (y, z)) \mapsto (x, z)$. The product is required to be associative, that is $(pq)r = p(qr)$ whenever these products are defined.

A \mathbf{C}^\times groupoid actually has two other important algebraic structures, an identity and inverse which could have been included in the definition but in fact are a consequence of it. The identity is a section e of the bundle P over the diagonal in X^2 which satisfies

$pe = ep = p$. To define it note first that if $p \in P_{(x,y)}$ and $q \in P_{(y,y)}$ then $pq \in P_{(x,y)}$. Hence there is some $z \in \mathbf{C}^\times$ such that $pq = pz$. Define $e = qz^{-1}$ so that $pe = p$. Because the product is a bundle automorphism $(pw)e = (pe)w = pw$ for all $w \in \mathbf{C}^\times$ and hence $qe = q$ for every q in P . To show that $ep = p$ we use associativity. Clearly $ep = pz$ for some $z \in \mathbf{C}^\times$ and considering $(pe)p = p(ep)$ it follows that $z = 1$. To define the inverse notice that the equation $pq = e$ can be solved by acting by \mathbf{C}^\times . Then we have that $qp = ez$ for some z and using associativity in the form of $p(qp) = (pq)p$ it follows that $pez = ep = pe$ and hence $z = 1$. The inverse will be denoted by $p \mapsto p^{-1}$. To understand the global structure of the inverse notice that it is possible to construct a bundle P^* over X^2 by defining it to be the same set as P but changing the \mathbf{C}^\times action to $pz = pz^{-1}$. If P is the frame bundle of a line bundle L then P^* is the frame bundle of L^* . Because \mathbf{C}^\times is abelian this is still a right action. The inversion then defines a map $P \rightarrow P^*$ covering the map $X^2 \rightarrow X^2$ defined by $(x, y) \mapsto (y, x)$. The identity and the inverse behave as one would expect with respect to the product.

Given a \mathbf{C}^\times groupoid we can recover the definition in terms of categories [2] by taking X as the set of objects and $P_{(x,y)}$ as the morphisms from x to y .

A simple example of a \mathbf{C}^\times groupoid is constructed by taking a \mathbf{C}^\times bundle Q on X and defining $P_{(x,y)} = \text{Aut}_{\mathbf{C}^\times}(Q_x, Q_y)$ where Q_x is the fibre of Q over x and the subscript \mathbf{C}^\times indicates that these automorphisms commute with the \mathbf{C}^\times action. An alternative way to define this is to use the two projections π_1 and π_2 on the first and second factors of $X^2 = X \times X$ and define $P = \pi_1^{-1}Q^* \otimes \pi_2^{-1}Q$. The composition is, of course, composition of automorphisms. What makes \mathbf{C}^\times groupoids uninteresting is that every \mathbf{C}^\times groupoid arises in this way! To see this let P be a \mathbf{C}^\times groupoid and pick a basepoint x in X . Then define a \mathbf{C}^\times bundle Q on X by $Q_y = P_{(x,y)}$, that is Q is the pull-back of P under the map $y \mapsto (x, y)$. Then the composition and inversion can be used to define a \mathbf{C}^\times bundle isomorphism

$$Q_y^* \otimes Q_z = P_{(x,y)}^* \otimes P_{(x,z)} \rightarrow P_{(y,z)} \quad (2.2)$$

by $(p, q) \mapsto p^{-1}q$. It is easy to see that this is a \mathbf{C}^\times bundle isomorphism and that, moreover, it preserves the composition. Hence it is an isomorphism of \mathbf{C}^\times groupoids.

Although we have just seen that the theory of \mathbf{C}^\times groupoids is nothing more than the theory of \mathbf{C}^\times bundles over pointed sets it is useful to develop the theory further as the next section of bundle gerbes is then a straightforward generalisation.

A connection ∇ on the bundle $P \rightarrow X^2$ gives rise to a connection on $P \otimes P$ and hence on $P \circ P$. Call it a groupoid connection if it is mapped by the product to itself again. Such connections exist because we can identify P with $\pi_1^{-1}Q^* \otimes \pi_2^{-1}Q$ for some $Q \rightarrow X$ and pick a connection on Q and pull it back to a connection on P . The curvature F_∇ of a

groupoid connection on P constructed in this way has the form

$$F_{\nabla} = \pi_2^*(f) - \pi_1^*(f) \quad (2.3)$$

where f is the curvature of the original connection on $Q \rightarrow X$. It is also possible to show (see Section 8 below) that if F_{∇} is the curvature of any groupoid connection then there is unique two-form f on X which satisfies (2.3). Call this two-form f the groupoid curvature.

It is well known that given an integral closed two-form $f/(2\pi i)$ over a 1-connected space X it is possible to explicitly construct a \mathbf{C}^\times bundle over X with a connection whose curvature is f . It will be useful later to know how to repeat this construction for \mathbf{C}^\times groupoids. In fact this construction is a little more natural in the groupoid context as, unlike the construction of the \mathbf{C}^\times bundle over X , it does not require the choice of a base point.

Assume then that X is a 1 connected manifold and that $f/(2\pi i)$ is a closed, integral two-form on X . We shall construct a \mathbf{C}^\times groupoid on X with a groupoid connection whose groupoid curvature is f . Let PX be the space of all piecewise smooth paths in X and let $PX_{(x,y)}$ be the set of all paths beginning at x and ending at y . We need piecewise smooth paths as we will want to compose them to define the groupoid product. Define an equivalence relation on $PX_{(x,y)} \times \mathbf{C}^\times$ by saying that (γ, z) is equivalent to $(\tilde{\gamma}, \tilde{z})$ if

$$z = \exp\left(\int_D f\right)\tilde{z} \quad (2.4)$$

where D is a map of the disk into X with boundary the union of γ and $\tilde{\gamma}$ and oriented by γ . Let $P_{(x,y)}$ denote the quotient space. The \mathbf{C}^\times action is just that induced by $(\gamma, z)w = (\gamma, zw)$. If $\gamma(1) = \tilde{\gamma}(0)$ then define a new piecewise smooth path $\gamma \star \tilde{\gamma}$ by running along γ at twice the speed for $t \in [0, 1/2]$ and then running along $\tilde{\gamma}$ at twice the speed for $t \in [1/2, 1]$. A product on $PX \times \mathbf{C}$ is then defined by $(\gamma, z) \star (\tilde{\gamma}, \tilde{z}) = (\gamma \star \tilde{\gamma}, z\tilde{z})$ and it is straightforward to check that this descends to a product on P making it a \mathbf{C}^\times groupoid.

Now we construct a \mathbf{C}^\times groupoid connection. Notice that P is the quotient of a fibering

$$PX \times \mathbf{C} \rightarrow P \quad (2.5)$$

where the fibres are defined by the equivalence relation (2.4). The tangent space to $PX \times \mathbf{C}$ at (γ, z) is the space of all pairs (ξ, α) where ξ is a vector field along γ and α is a complex number. A vector field along γ means a continuous vector field which is smooth when γ is smooth. The subspace of the tangent space that is tangent to the fibering (2.5) is

$$K_{(\gamma, z)} = \{(\xi, \alpha) \mid \alpha = -\int_0^1 f(\gamma', \xi) dt, \xi(0) = 0 = \xi(1)\} \quad (2.6)$$

where γ' is the tangent vector field along γ . Consider the map

$$\text{ev}: PX \times [0, 1] \rightarrow X \quad (2.7)$$

which maps $(\gamma, t) \mapsto \gamma(t)$. A one-form \hat{A} on $PX \times [0, 1]$ is now defined by pulling back f and integrating it over the $[0, 1]$ direction and then letting

$$\hat{A} = dt + \int_0^1 \text{ev}^*(f). \quad (2.8)$$

Notice that

$$d\hat{A} = \text{ev}_1^*(f) - \text{ev}_0^*(f) \quad (2.9)$$

where $\text{ev}_t(\gamma) = \gamma(t)$. The forms \hat{A} and $d\hat{A}$ both annihilate vectors in the space (2.6) and hence \hat{A} descends to a one-form A on P which defines a connection.

It remains to check that this connection is a groupoid connection. The product map defines a sum on tangent vectors. If (ξ, α) , and $(\tilde{\xi}, \tilde{\alpha})$ are tangent to (γ, z) and $(\tilde{\gamma}, \tilde{z})$ then $(\xi \star \tilde{\xi}, \alpha + \tilde{\alpha})$ is tangent at $(\gamma, z) \star (\tilde{\gamma}, \tilde{z})$ where $\xi \star \tilde{\xi}$ is the obvious vector field along $\gamma \star \tilde{\gamma}$. The essential point in the proof that A preserves the product is that

$$\int_0^1 f(\gamma', \xi) dt + \int_0^1 f(\tilde{\gamma}', \tilde{\xi}) dt = \int_0^1 f((\gamma \star \tilde{\gamma})', \xi \star \tilde{\xi}) dt. \quad (2.10)$$

It follows from (2.9) that

$$dA = \pi_2^*(f) - \pi_1^*(f) \quad (2.11)$$

and hence the curvature of this groupoid connection is f .

In the next section we are concerned with bundle gerbes which are fibrations whose fibres are \mathbf{C}^\times groupoids. Then it may not be possible to choose basepoints continuously and the constructions above become more interesting.

3. \mathbf{C}^\times bundle gerbes

Consider a fibration $\pi: Y \rightarrow M$. Define the fibre product $Y^{[2]}$ in the usual way, that is the subset of pairs (y, y') in $Y \times Y$ such that $\pi(y) = \pi(y')$. Notice that the diagonal is inside $Y^{[2]}$ and that the map that transposes elements of $Y^2 = Y \times Y$ fixes $Y^{[2]}$. Denote by π_i the restriction of the projection maps on Y^2 to $Y^{[2]}$. Denote by $Y^{[2]} \circ Y^{[2]}$ the intersection of $Y^{[2]} \times Y^{[2]}$ with $Y^2 \circ Y^2$. If P and Q are \mathbf{C}^\times bundles over $Y^{[2]}$ denote by $P \circ Q$ the restriction to $Y^{[2]} \circ Y^{[2]}$ of the bundle $\pi_1^{-1}(P) \otimes \pi_2^{-1}(Q)$ over $Y^{[2]} \times Y^{[2]}$.

A bundle gerbe over M is defined to be a choice of a fibration $Y \rightarrow M$ and a \mathbf{C}^\times bundle $P \rightarrow Y^{[2]}$ with a product, that is, a \mathbf{C}^\times bundle isomorphism $P \circ P \rightarrow P$ covering $((y_1, y_2), (y_2, y_3)) \mapsto (y_1, y_3)$. The product is required to be associative whenever triple

products are defined. Just as for \mathbf{C}^\times groupoids a bundle gerbe has an inverse and an identity denoted by the same symbols. Occasionally we shall denote a bundle gerbe as a triple (P, Y, M) .

Example: 3.1 Let $Q \rightarrow Y$ be a principal \mathbf{C}^\times bundle. Define $P_{(x,y)} = \text{Aut}_{\mathbf{C}^\times}(Q_x, Q_y) = Q_x^* \otimes Q_y$. Then this defines a bundle gerbe called the trivial bundle gerbe. We also have $P = \text{Aut}(\pi_1^{-1}Q, \pi_2^{-1}Q) = \pi_1^{-1}Q^* \otimes \pi_2^{-1}Q$.

Example: 3.2 An example of interest in [1] is to start with a fibration $Y \rightarrow M$ with 1 connected fibres and a two-form on Y whose restriction to each fibre is closed and integral. Then we can apply the construction in Section 2 fibre by fibre to define a bundle gerbe $P \rightarrow Y^{[2]}$.

A morphism of bundle gerbes (P, Y, M) and (Q, X, N) is a triple of maps (α, β, γ) . The map $\beta: Y \rightarrow X$ is required to be a morphism of the fibrations $Y \rightarrow M$ and $X \rightarrow N$ covering $\gamma: M \rightarrow N$. It therefore induces a morphism $\beta^{[2]}$ of the fibrations $Y^{[2]} \rightarrow M$ and $X^{[2]} \rightarrow N$. The map α is required to be a morphism of \mathbf{C}^\times bundles covering $\beta^{[2]}$ which commutes with the product and hence also with the identity and inverse. A morphism of bundle gerbes over M is a morphism of bundle gerbes for which $M = N$ and γ is the identity on M .

Various constructions are possible with bundle gerbes. We can define a pull-back and product as follows. If (Q, X, N) is a bundle gerbe and $f: M \rightarrow N$ is a map then we can pull back the fibration $X \rightarrow N$ to obtain a fibration $f^{-1}(X) \rightarrow M$ and a morphism of fibrations $f^{-1}: f^{-1}(X) \rightarrow X$ covering f . This induces a morphism $(f^{-1}(X))^{[2]} \rightarrow X^{[2]}$ and hence we can use this to pull back the \mathbf{C}^\times bundle Q to a \mathbf{C}^\times bundle $f^{-1}(Q)$ say on $f^{-1}(X)$. It is easy to check that $(f^{-1}(Q), f^{-1}(X), M)$ is a bundle gerbe, the pull-back by f of the gerbe (Q, X, N) . If (P, Y, M) and (Q, X, M) are bundle gerbes over M then we can form a fibre product $Y \times_M X \rightarrow M$ and then form a \mathbf{C}^\times bundle $P \otimes Q$ over $(Y \times_M X)^{[2]}$ which is the product of the bundle gerbes (P, Y, M) and (Q, X, M) .

Notice that, unlike the case of \mathbf{C}^\times groupoids, it is not clear that every bundle gerbe is trivial. The proof in Section 2 that a groupoid is the same as a bundle over a pointed set can only be applied fibre by fibre if $Y \rightarrow M$ has a section. We shall see in Section 5 that there is associated to a bundle gerbe a class in $H^3(M, \mathbf{Z})$, its Dixmier-Douady class, which is precisely the obstruction to the bundle gerbe being trivial.

4. Central extensions

A motivating example of a bundle gerbe is the bundle gerbe arising from a central extension of groups. Let

$$0 \rightarrow \mathbf{C}^\times \xrightarrow{\iota} \hat{G} \xrightarrow{p} G \rightarrow 0 \quad (4.1)$$

be a central extension of groups and $Y \rightarrow M$ a principal G bundle. Then it may happen that there is a principal \hat{G} bundle \hat{Y} and a bundle map $\hat{Y} \rightarrow Y$ commuting with the homomorphism $\hat{G} \rightarrow G$. In such a case Y is said to lift to a \hat{G} bundle. One way of answering the question of when Y lifts to a \hat{G} bundle is to present Y with transition functions $g_{\alpha\beta}$ relative to a cover $\{U_\alpha\}$ of M . If the cover is sufficiently nice we can lift the $g_{\alpha\beta}$ to maps $\hat{g}_{\alpha\beta}$ taking values in \hat{G} and such that $p(\hat{g}_{\alpha\beta}) = g_{\alpha\beta}$. These are candidate transition functions for a lifted bundle \hat{Y} . However they may not satisfy the cocycle condition $\hat{g}_{\alpha\beta}\hat{g}_{\beta\gamma}\hat{g}_{\gamma\alpha} = 1$ and indeed there is a \mathbf{C}^\times valued map

$$e_{\alpha\beta\gamma}: U_\alpha \cap U_\beta \cap U_\gamma \rightarrow \mathbf{C}^\times \quad (4.2)$$

defined by $\iota(e_{\alpha\beta\gamma}) = \hat{g}_{\alpha\beta}\hat{g}_{\beta\gamma}\hat{g}_{\gamma\alpha}$. Because (4.1) is a central extension it follows that $e_{\alpha\beta\gamma}$ is a co-cycle and hence defines a class in $H^2(M, \mathbf{C}^\times)$. It is well-known that the coboundary map in the long exact sequence of cohomology induced by (4.1) defines an isomorphism

$$H^2(M, \mathbf{C}^\times) \simeq H^3(M, \mathbf{Z}). \quad (4.3)$$

The image of $e_{\alpha\beta\gamma}$ under this coboundary is the class in $H^3(M, \mathbf{Z})$ which is the obstruction to $Y \rightarrow M$ lifting to \hat{G} .

Another way to see when $Y \rightarrow M$ lifts to \hat{G} is to construct a bundle gerbe which is trivial precisely when a lift is possible. To do this define $P \rightarrow Y^{[2]}$ by

$$\hat{P}_{(x,y)} = \{h \in \hat{H} \mid xp(h) = y\}. \quad (4.4)$$

Assume now that Y has a lift to a principal \hat{G} bundle \hat{Y} over M so that there is a projection $q: \hat{Y} \rightarrow Y$ commuting with p in the appropriate way. Then $\hat{Y} \rightarrow Y$ is a \mathbf{C}^\times bundle over Y . Indeed let $g \in P_{(x,y)}$; then it defines a map $\hat{Y}_x \rightarrow \hat{Y}_y$ which, by centrality, commutes with the \mathbf{C}^\times action. This defines an isomorphism

$$P_{(x,y)} \simeq \text{Aut}_{\mathbf{C}^\times}(\hat{Y}_x, \hat{Y}_y) \quad (4.5)$$

so that $P \simeq (\pi_1^* \hat{Y})^* \otimes \pi_2^* \hat{Y}$. On the other hand if the bundle gerbe P is trivial, say isomorphic to $\text{Aut}(\hat{Y}, \hat{Y})$ for some \mathbf{C}^\times bundle $\hat{Y} \rightarrow Y$ it is possible to define an action of \hat{G} on \hat{Y} and make it a lift of Y . To do this start with g in \hat{G} and the fibre \hat{Y}_y . Then define x by $xp(g) = y$. Then $g \in P_{(x,y)} = \text{Aut}_{\mathbf{C}^\times}(\hat{Y}_x, \hat{Y}_y)$ so apply the corresponding automorphism to any element in \hat{Y}_x to define the action of g . It can be checked that this defines a lift of Y .

This proves that Y lifts to \hat{G} if and only if the gerbe P is trivial. In other words the bundle gerbe P is trivial when the three class defined by Y is zero. We shall see in the next

section that this examples generalises to all bundle gerbes. Every bundle gerbe defines a three class, its Dixmier-Douady class, which is the obstruction to it being trivial.

5. The Dixmier-Douady class of a bundle gerbe.

Let $P \rightarrow Y^{[2]}$ be a bundle gerb. Choose a cover $\{U_\alpha\}$ of M such that over each U_α there is a section s_α of Y . Then on the overlap $U_\alpha \cap U_\beta$ we have a map

$$(s_\alpha, s_\beta): U_\alpha \cap U_\beta \rightarrow Y^{[2]} \quad (5.1)$$

defined by $(s_\alpha, s_\beta)(x) = (s_\alpha(x), s_\beta(x))$. Let $P_{\alpha\beta}$ be the pull-back of P via this map. Notice that the product gives an isomorphism $P_{\alpha\beta} \otimes P_{\beta\gamma} \simeq P_{\alpha\gamma}$. Choose sections $\sigma_{\alpha\beta}$ of each $P_{\alpha\beta}$. Then using the product we define a \mathbf{C}^\times valued function $g_{\alpha\beta\gamma}$ defined by

$$\sigma_{\alpha\beta}\sigma_{\beta\gamma} = \sigma_{\alpha\gamma}g_{\alpha\beta\gamma}. \quad (5.2)$$

It is easy to check that g defines a class in $H^2(M, \mathbf{C}^\times)$. The image of g under this the isomorphism (4.3) is called the Dixmier-Douady class of the bundle gerbe P .

I claim that the Dixmier-Douady class is precisely the obstruction to a gerbe being trivial. To see this note first that if P trivial, say $P = \pi_1^{-1}Q^* \otimes \pi_2^{-2}Q$ for some bundle Q on Y we can define $Q_\alpha = s_\alpha^*(Q)$ and we then have canonical isomorphisms

$$P_{\alpha\beta} = Q_\alpha^* \otimes Q_\beta \quad (5.3)$$

commuting with products. Hence in the construction of the cocycle in Section 4, if we choose δ_α to be a section of Q_α and define $\sigma_{\alpha\beta} = (\delta_\alpha)^{-1} \otimes \delta_\beta$ we obtain a trivial cocycle g .

If on the other hand g is trivial, say

$$g_{\alpha\beta\gamma} = \rho_{\alpha\beta}\rho_{\beta\gamma}\rho_{\gamma\alpha} \quad (5.5)$$

where ρ is \mathbf{C}^\times valued, then we can divide $\sigma_{\alpha\beta}$ in equation (5.2) by $\rho_{\alpha\beta}$ and hence without loss of generality assume that g is identically one. Let $Y_\alpha = \pi^{-1}(U_\alpha)$. Define a principal bundle Q_α over Y_α by defining its fibre at y to be

$$(Q_\alpha)_y = P_{(y, s_\alpha(\pi(y)))}. \quad (5.6)$$

The $\sigma_{\alpha\beta}$ are elements of

$$P_{(s_\alpha(\pi(y)), s_\beta(\pi(y)))} = P_{s_\alpha(\pi(y)), y}^* \otimes P_{s_\alpha(\pi(y)), y} = (Q_\alpha)_y^* \otimes (Q_\beta)_y. \quad (5.7)$$

The $\sigma_{\alpha\beta}$ therefore define automorphisms between Q_α and Q_β over $Y_\alpha \cap Y_\beta$. The standard clutching construction now defines a bundle Q over all of Y and it is straightforward to check that this trivialises the gerbe P over Y .

Example 5.1: If the fibration $Y \rightarrow M$ admits a global section then we can smoothly pick a base point in each fibre. The results of Section 2 can then be applied to show that the bundle gerbe is trivial. It is a trivial consequence of the definition in Section 4 that the Dixmier-Douady class is zero.

The Dixmier-Douady class behaves naturally with respect to the operations on bundle gerbes defined in section 3. The Dixmier-Douady class of a pull-back bundle gerbe is the pull-back of the Dixmier-Douady class and the Dixmier-Douady class of a product of two bundle gerbes is the product (sum) of the Dixmier-Douady classes of the individual bundle gerbes.

6. Connections on bundle gerbes.

Consider a connection ∇ on a bundle gerbe $P \rightarrow Y^{[2]}$. Then it induces a connection $\nabla \circ \nabla$ on the bundle $P \circ P \rightarrow Y^{[2]} \circ Y^{[2]}$. The connection ∇ is said to be a bundle gerbe connection if the image of $\nabla \circ \nabla$ under the product map is ∇ .

It is not clear that bundle gerbe connections exist. To see that they do note that if the bundle gerbe is trivial i.e. $P = \text{Aut}(\pi_1^{-1}Q, \pi_2^{-1}Q)$ for some principal bundle Q on Y then a connection ∇ on Q defines a bundle gerbe connection $\nabla^* \otimes \nabla$ on P . Now choose an open cover $\{U_\alpha\}$ of M over which the fibration $Y \rightarrow M$ is trivial. Denote by Y_α the open subset of Y which is the pre-image under the projection of U_α . Then the bundle gerbe can be trivalised over Y_α and hence admits a bundle gerbe connection. Choose a partition of unity for the open cover $\{U_\alpha\}$ of M . This pulls-back to $Y^{[2]}$ to give a partition of unity for the open cover $\{Y_\alpha^{[2]}\}$ and can be used to patch together the bundle gerbe connections on the various open sets to give a bundle gerbe connection on P . It will follow from the results in Section 8 that the space of all bundle gerbe connections is an affine space for the vector space $\Omega^1(Y)/\pi^*(\Omega^1(M))$ of all one-forms on Y modulo one-forms pulled back from M .

7. The curvature of a bundle gerbe connection.

A bundle gerbe connection ∇ is a connection so it has a curvature F_∇ which is a two-form on $Y^{[2]}$. In the case that this is a trivial gerbe and the connection is the tensor product connection then the curvature can be written as

$$F_\nabla = \pi_2^*f - \pi_1^*f \tag{7.1}$$

where f is the curvature of the connection on the bundle over Y and the π_i are the two projections $\pi_i: Y^{[2]} \rightarrow Y$. We shall see in the next section that it is always possible to find an f satisfying equation (7.1). This is certainly true for a bundle gerbe connection constructed by a partition of unity argument as in Section 6. The choice of such an f we will call a curvature for the gerbe connection.

Consider now df . In the case that this is a trivial gerbe we have, of course, that $df = 0$. More generally we have $dF_\nabla = 0$ so that

$$\pi_1^* df = \pi_2^* df. \quad (7.2)$$

I claim that this means that $df = \pi^*(\omega)$ for some three-form ω on M . To see this note that a point of $Y^{[2]}$ is a pair (x, z) where $\pi(x) = \pi(z)$ and a tangent vector to $Y^{[2]}$ at (x, z) is a pair (X, Z) with $X \in T_x Y$ and $Z \in T_z Y$ and $\pi_*(X) = \pi_*(Z)$. Then equation (7.2) says that

$$df(x)(X_1, X_2, X_3) = df(z)(Z_1, Z_2, Z_3) \quad (7.3)$$

whenever $\pi_*(X_i) = \pi_*(Z_i)$ for $i = 1, 2, 3$. Hence if $m \in M$ and $\xi_i \in T_m M$ choose $x \in Y$ and $X_i \in T_x Y$ such that $\pi(x) = m$ and $\pi_*(X_i) = \xi_i$ and define

$$\omega(m)(\xi_1, \xi_2, \xi_3) = df(x)(X_1, X_2, X_3). \quad (7.4)$$

Equation (7.3) shows that this definition is independent of the choice of x and the X_i . Clearly $\pi^*(\omega) = df$ and moreover ω is closed. We call $\omega/(2\pi i)$ the Dixmier-Douady form of the pair (∇, f) .

It will follow from the results in Section 8 that the various choices in this construction do not change the cohomology class of the Dixmier-Douady form. We shall see in Section 11 that the de Rham cohomology class of the Dixmier-Douady form is the image in real cohomology of the Dixmier-Douady class defined in Section 5. This proves in particular that the Dixmier-Douady form is integral a fact that also follows from the discussion of holonomy in Section 12.

8. A complex with no cohomology

For a fibration Y let $Y^{[p]}$ denote the p th fibered product. There are projection maps $\pi_i: Y^{[p]} \rightarrow Y^{[p-1]}$ which omit the i th element for each $i = 1 \dots p$. These define a map

$$\delta: \Omega^q(Y^{[p-1]}) \rightarrow \Omega^q(Y^{[p]}) \quad (8.1)$$

by

$$\delta(\omega) = \sum_{i=1}^p (-1)^i \pi_i^*(\omega). \quad (8.2)$$

Clearly $\delta^2 = 0$ so that $\Omega^q(Y^{[*]})$ is a complex. We wish to show that this complex has no cohomology. This will then settle the question of the existence of f in Section 8 as we have $F_\nabla \in \Omega^2(Y^{[2]})$ with $\delta(F_\nabla) = 0$ and hence $F_\nabla = \delta(f)$ for some $f \in \Omega^2(Y^{[1]}) = \Omega^2(Y)$.

Consider first the case that the fibration is trivial, say $Y = M \times F$. The general case will follow by a partition of unity argument. In this case $Y^{[p]} = M \times F^p$. Because the

notation is cumbersome at this point it will be convenient to denote a collection of q vectors (X^1, \dots, X^q) just by X and the action of a q form τ on these vectors by $\tau(X)$ rather than $\tau(X^1, \dots, X^q)$. When we are dealing with vectors tangent to F^{p+1} at $f = (f_1, \dots, f_{p+1})$ then each of the X is a collection of vectors (X_1, \dots, X_{p+1}) where each X_j is a q -tuple of vectors in $T_{f_j}(F)$. So, with these notational conventions we have for $\omega \in \Omega^q(Y^{[p]})$

$$\begin{aligned} \delta(\omega)(m, f)(\xi, (X_1, \dots, X_{p+1})) \\ = \sum_i (-1)^i \omega(m, f_1, \dots, \hat{f}_i, \dots, f_p)((\xi, (X_1, \dots, \hat{X}_i, \dots, X_{p+1}))) \end{aligned} \quad (8.3)$$

where ξ is a q -tuple of vector tangent to M at m . Now fix a point f in F and a q tuple of vectors X in the tangent space at f and define $\rho \in \Omega^q(Y^{[p-1]})$ by

$$\rho(m, f_1, \dots, f_p)(\xi, X_1, \dots, X_p) = \omega(m, f_1, \dots, f_p, f)(\xi, X_1, \dots, X_p, X). \quad (8.4)$$

It follows from the fact that $\delta(\omega) = 0$ that $\delta(\rho) = (-1)^{p+1}\omega$. This proves the required result in the case that Y is the trivial fibration. The general case is proved by choosing an open cover U_α such that Y is trivial over each U_α . Then let ψ_α be a partition of unity subordinate to that cover. Let Y_α be the part of Y sitting over U_α and similarly for $(Y^{[p]})_\alpha$. Note that $(Y^{[p]})_\alpha = (Y_\alpha)^{[p]}$. There are projection maps for each $Y^{[p]} \rightarrow M$ and we can pull the partition of unity back to any of these spaces. We will denote it by the same symbol. If we start with ω in $\Omega^q(Y^{[p]})$ then $\omega|_{U_\alpha} = \delta(\rho_\alpha)$ for some ρ_α . Hence we have

$$\omega = \sum_\alpha \psi_\alpha \delta(\rho_\alpha) = \delta\left(\sum_\alpha \psi_\alpha \rho_\alpha\right) = \delta(\rho) \quad (8.5)$$

where $\rho = \sum_\alpha \psi_\alpha \rho$.

At $p = 1$ we define $Y^{[0]} = M$ and let $\delta: \Omega^q(M) \rightarrow \Omega^q(Y)$ be pull-back under π . Exactness follows exactly as for the proof, in Section 7, that there exists an ω such that $\pi^*(\omega) = df$.

We can now confirm two facts stated earlier. The first is the affine space structure of the space of all bundle gerbe connections. If ∇ and ∇' are two bundle gerbe connections they clearly differ by a one-form η on $Y^{[2]}$ with $\delta(\eta) = 0$. So $\eta = \delta(\mu)$. On the other hand $\nabla + \delta(\mu)$ is a bundle gerbe connection for any μ on Y and $\delta(\mu) = 0$ for such a μ precisely when μ is pulled back from M . This gives the required result. The second fact is the independence of the class of the Dixmier-Douady form from various choices. The first is the choice of f satisfying $\delta(f) = F_\nabla$. If f' is another such then $\delta(f - f') = 0$ and hence $f - f' = \pi^*(\rho)$ so that $df - df' = \pi^*(\rho)$ and $\omega - \omega' = d\rho$. The other choice is

the choice of bundle gerbe connection. If we have $\nabla' = \nabla + \delta(\mu)$ then we can choose f' so that $f = f' + d\mu$ and hence $df = df'$ so that $\omega = \omega'$.

9. Gerbes, connective structures and curvings

The relationship with the theory of gerbes discussed in [1] is as follows. For any open set $U \subset M$ let $C(U)$ be the set of all sections of Y which we want to think of as the objects in a category, in fact in a groupoid. If s and t are two such sections they define a section (s, t) of $Y^{[2]}$ over U by $m \mapsto (s(m), t(m))$. The morphisms from s to t we define to be the sections of the bundle $(s, t)^{-1}P$ over U . The composition is constructed from the composition on P . This construction defines a pre-sheaf of groupoids. The sheafification of this presheaf gives rise to a gerbe in the sense of Brylinski [1].

In [1] Brylinski introduces the notion of connective structure and curving. We indicate here how these are related to the bundle gerbe connection and its curvature. Let ∇ be a bundle gerbe connection for the bundle gerbe P over $Y \rightarrow M$. Let U be an open subset of M over which Y admits a section $s: U \rightarrow Y$. Denote by \hat{s} the induced map $Y|_U \rightarrow (Y|_U)^{[2]}$ defined by $y \mapsto (y, s(\pi(y)))$. Then we have an isomorphism $P \simeq \hat{s}^{-1}P \otimes \hat{s}^{-1}P^*$ defined by the product

$$P_{(p,q)} \rightarrow P_{(s(\pi(p)),p)}^* \otimes P_{(s(\pi(p)),q)} \quad (9.1)$$

which trivialises P over $Y|_U \rightarrow U$. Consider the set $Co(s)$ of all connections A on the bundle $\hat{s}^{-1}P$ such that $\delta(A) = \nabla$. This space of connections is an affine space for Ω_U^1 the space of all 1-forms on U and hence $Co(s)$ defines a Ω_U^1 torsor. This torsor is a connective structure in the sense of Brylinski.

Assume now that we have chosen a two-form f on Y such that $\delta(f) = F_\nabla$ where F_∇ is the curvature of ∇ . Then to any A in $Co(P)$ we can define a two-form $K(A)$ on U by

$$\pi^*(K(A)) = F_A - f \quad (9.2)$$

where F_A is the curvature of A . This equation makes sense because

$$\delta(F_A - f) = F_{\delta(A)} - \delta(f) = F_\nabla - F_\nabla = 0. \quad (9.3)$$

Finally notice that $\pi^*(dK(A)) = -df$ so that $dK(A) = -\omega$ so that, up to sign, this is the curvature of the bundle gerbe.

The definition of morphism of bundle gerbes on M in section 3 naturally gives rise to a notion of isomorphism. A fundamental result about gerbes is the theorem that the Dixmier-Douady class gives an exact correspondence between elements of $H^3(M, \mathbf{Z})$ and equivalence classes of gerbes [1]. This is not true for bundle gerbes on M and bundle gerbe isomorphism; there are bundle gerbes on M which are not isomorphic but which have the

same Dixmier-Douady class and hence define equivalent gerbes. Indeed it is not hard to show that if $(\alpha, \beta, 1_M)$ is a morphism of bundle gerbes (P, Y, M) and (Q, X, M) then the Dixmier-Douady classes of (P, Y, M) and (Q, X, M) are the same. In this example X and Y can be quite different. For example if Y admits a global section we can take X to be the image of that section and Q the restriction of P to X . A bundle gerbe where the fibers are points clearly has Dixmier-Douady class zero and we have already seen that a bundle gerbe where the fibration has a section also has Dixmier-Douady class zero. This dependence of bundle gerbes on the choice of a fibration is nicely eliminated by the gerbe concept.

10. The tautological bundle gerbe

Let $\omega/(2\pi i)$ be a form representing a class in $H^3(M, \mathbf{Z})$ where M is 2 connected. We shall show how to construct a fibration of groupoids with ω as its curvature. Recall that if Σ is an oriented two sphere in M the Wess Zumino Witten action is an element of \mathbf{C}^\times associated to Σ by extending Σ to a ball B in M and defining

$$\text{wzw}(\Sigma) = \exp\left(\int_B \omega\right). \quad (10.1)$$

Similarly if Σ and Σ' are two disks in M with common boundary denote by $\text{wzw}(\Sigma, \Sigma')$ the Wess Zumino Witten action of the sphere formed by their union if it is given the orientation of the first disk.

Fix a base point for M and let $Y \rightarrow M$ be the path-fibration. Then $Y^{[2]}$ consists of all pairs of paths beginning at the basepoint and with the same endpoints. Define the fibre of P at such a point by taking all pairs consisting of a piecewise smooth surface with these two paths as boundary and a non-zero complex number and defining an equivalence relation

$$(\Sigma, z) \sim (\Sigma', z') \quad (10.1)$$

if $z = \text{wzw}(\Sigma, \Sigma')z'$. Denote equivalence classes by square-brackets. Then the set of all equivalence classes forms a principal \mathbf{C}^\times bundle over Y . We need to show that it is a bundle gerbe by constructing a product.

The product map $P_{(x,y)} \otimes P_{(y,z)} \rightarrow P_{(x,z)}$ is defined by

$$[\Sigma, z] \otimes [\Sigma', z'] \rightarrow [\Sigma \cup \Sigma', zz']. \quad (10.2)$$

This makes sense because Σ and Σ' have half of each of their boundaries (the curve y) in common.

We now show that this bundle gerbe has a bundle gerbe connection whose Dixmier-Douady form is $\omega/(2\pi i)$. We could perform calculations analogous to those in Section 2

however it is simpler to actually use those calculations as follows. Consider the evaluation map

$$\text{ev}: Y \times [0, 1] \rightarrow M \quad (10.3)$$

and use it to define a closed two-form $f = \int_0^1 \text{ev}^*(\omega)$. Note that $f/(2\pi i)$ is integral. We can now repeat the constructions in Section 2 but restrict them to $Y^{[2]} \circ Y^{[2]} \subset Y^{[2]} \times Y^{[2]}$. This defines the bundle gerbe with connection ∇ and curvature F_∇ satisfying $F_\nabla = \pi_2^*(f) - \pi_1^*(f)$. It is now an easy calculation to show that if $\pi: Y \rightarrow M$ then $df = \pi^*(\omega)$ as required.

11. Deligne cohomology

The Deligne cohomology of M that we are interested in is the total cohomology of the log-complex

$$0 \rightarrow C^\infty(M)^\times \rightarrow \Omega^1(M) \rightarrow \dots \rightarrow \Omega^p(M) \rightarrow 0. \quad (11.1)$$

Here the first non-zero map is the exterior derivative of the log or $f \mapsto df/f$. If $p = 1$ then the elements of H^1 of this total cohomology are represented in Čech cohomology with respect to an open cover by pairs $(A_\alpha, \sigma_{\alpha\beta})$ subject to the condition that

$$A_\alpha - A_\beta = \sigma_{\alpha\beta}^{-1} d\sigma_{\alpha\beta}. \quad (11.2)$$

It is not hard to show that the elements of this cohomology are equivalence classes of \mathbf{C}^\times bundles with connection.

We shall show that in the case $p = 2$ that we can manufacture a class in this total cohomology from a gerbe with connection and curvature. A class in this cohomology will be a triple

$$(f_\alpha, A_{\alpha\beta}, g_{\alpha\beta\gamma}). \quad (11.3)$$

These have to satisfy

$$A_{\alpha\beta} - A_{\beta\gamma} + A_{\gamma\alpha} = g_{\alpha\beta\gamma}^{-1} dg_{\alpha\beta\gamma} \quad (11.4)$$

and

$$f_\alpha - f_\beta = dA_{\alpha\beta}. \quad (11.5)$$

We have already a candidate for $g_{\alpha\beta\gamma}$. For the other two we first let $\nabla_{\alpha\beta}$ be the pull-back connection

$$\nabla_{\alpha\beta} = (s_\alpha, s_\beta)^* \nabla \quad (11.6)$$

and then define

$$A_{\alpha\beta} = \sigma_{\alpha\beta}^*(\nabla_{\alpha\beta}). \quad (11.7)$$

We also define

$$f_\alpha = s_\alpha^* f. \quad (11.8)$$

The first relation follows from the fact that a bundle gerbe connection preserves the product. So

$$\nabla_{\alpha\beta} \otimes \nabla_{\beta\gamma} = \nabla_{\alpha\gamma}. \quad (11.9)$$

But the pull-back of $\nabla_{\alpha\beta} \otimes \nabla_{\beta\gamma}$ with $\sigma_{\alpha\beta} \otimes \sigma_{\beta\gamma}$ is

$$A_{\alpha\beta} + A_{\beta\gamma}. \quad (11.10)$$

On the other hand this is also the pull-back of $A_{\alpha\gamma}$ with $\sigma_{\alpha\gamma} g_{\alpha\beta\gamma}$ or

$$A_{\alpha\gamma} + g_{\alpha\beta\gamma}^{-1} dg_{\alpha\beta\gamma} \quad (11.11)$$

and the result follows.

The second relation follows from the equation

$$F = \pi_1^* f - \pi_2^* f \quad (11.12)$$

by pulling-back both sides with (s_α, s_β) . On the LHS we get $dA_{\alpha\beta}$ and on the RHS $f_\alpha - f_\beta$.

It now follows readily that df_α is just the restriction of ω to U_α and moreover by standard double complex arguments it also follows that the class defined by $\omega/(2\pi i)$ is the same as the class defined by $g_{\alpha\beta\gamma}$. So the class defined by the Dixmier-Douady form is the image in $H^3(M, \mathbf{R})$ of the Dixmier-Douady class in $H^3(M, \mathbf{Z})$.

12. Holonomy of a bundle gerbe connection over a two-sphere

If we calculate the Deligne cohomology for a bundle gerbe with connection and curvature whose base-manifold is a two-sphere we can show that it is $U(1)$. The resulting number is a generalisation of holonomy. A simple way of understanding what this is is to first consider the case that the base manifold is not a two-sphere but in fact has $\pi_2(M) = 0$. In that case consider an embedded two-sphere Σ in M which is the boundary of some ball B in Σ . If $\omega/(2\pi i)$ is the Dixmier-Douady class then

$$\text{wzw}(\Sigma) = \exp\left(\int_B \omega\right) \quad (12.1)$$

is the holonomy of the bundle gerbe connection over Σ . In general this is not a satisfactory solution as it is analogous to it is like defining holonomy for a connection on a \mathbf{C}^\times bundle by integrating curvature over spanning disks. In general we want an answer intrinsic to the two-sphere in question.

To define such the holonomy choose a point in the two-sphere Σ and think of it as a disk with boundary identified. It is possible to lift this disk to a disk D in Y whose boundary lies entirely in one fibre. If we fix a point y in this fibre we define a map of the fibre into $Y^{[2]}$ by $y' \mapsto (y', y)$. We can then calculate the holonomy of the connection on P around the image of the boundary of D under this map. Call this $\text{hol}(\partial D, \nabla)$. In addition we can integrate f over D and form

$$\text{hol}(\partial D, \nabla)^{-1} \exp\left(\int_D f\right). \quad (12.2)$$

We need to show that this is independent of various choices. Let us leave the base point fixed for a moment and consider a second lift D' . Then we can define a disk \tilde{D} inside $Y^{[2]}$ by

$$\tilde{D} = \{(x, y) \mid x \in D, y \in D'\}. \quad (12.3)$$

From the elementary properties of holonomy we have

$$1 = \text{hol}(\partial \tilde{D}, \nabla)^{-1} \exp\left(\int_{\tilde{D}} F_{\nabla}\right) \quad (12.4)$$

and from the equation satisfied by f , (7.1) we conclude that

$$\text{hol}(\partial D, \nabla)^{-1} \exp\left(\int_D f\right) = \text{hol}(\partial D', \nabla)^{-1} \exp\left(\int_{D'} f\right). \quad (12.5)$$

To show that the holonomy is independent of the base point is a similar type of calculation but more involved. Let D_1 and D_2 be two lifts of Σ with different basepoints. Let D_3 denote the map of the cylinder into Y which covers Σ with the two basepoints removed and at each basepoint is coincides with either the boundary of D_1 of D_2 . Now by considering each pair of D_i respectively we define subsets of $Y^{[2]}$ by

$$D_{ij} = \{(x, y) \mid x \in D_i, y \in D_j\}. \quad (12.6)$$

Notice that topologically the union of these three cylinders is a cylinder with each end in the diagonal inside $Y^{[2]}$. On the diagonal the connection ∇ is flat so it follows that the integral of curvature F over the union of the D_{ij} is $2\pi i$ times an integer. Expanding out this integral as before gives the required result.

It is straightforward to check that if B is a three ball in M then we have

$$\text{hol}(\partial B, \nabla) = \exp\left(\int_B \omega\right) \quad (12.7)$$

We can use (12.7) to prove that the Dixmier-Douady form is integral. Indeed if $X \subset M$ is any three dimensional submanifold of M consider a family of three balls B_r inside X

shrinking to a point as $r \rightarrow 0$. Then the integral of ω over $X - B_r$ is the holonomy over the boundary of B_r but as B_r shrinks to a point this has limit 1 and hence the exponential of the integral of ω over all of X is 1. So the Dixmier-Douady form, $\omega/(2\pi i)$, is integral.

13. Topology of bundle gerbes

So far we have ignored any properties that the fibering $Y \rightarrow M$ must satisfy for there to be a non-trivial bundle gerbe $P \rightarrow Y$ ^[2]. However the examples we have considered all have $Y \rightarrow M$ having infinite dimensional fibres and we shall show now that this is a necessary condition. We use the result from [3] that if $Y \rightarrow M$ does not have infinite dimensional fibres then any smooth choice of closed p -form on the fibres is the restriction of a closed p -form on Y . If this is true then consider the two-form f on Y . Its restriction to each fibre is closed and hence by the theorem in [3] there exists a closed two-form ρ on Y such that $f - \rho$ is vertical. But $f - \rho$ and $d(f - \rho)$ are both vertical so we can find a two-form μ on M such that $\pi^*(\mu) = f - \rho$. Finally $\pi^*(\omega) = df = df - d\rho = \pi^*(d\mu)$ so that $\omega - d\mu$ and the bundle gerbe has trivial Dixmier-Douady class.

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